Algorithms for Transformation into the Extended Jordan Controllable and Observable Forms

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Introduction

Consider the following system

\[
\begin{align*}
\dot{x}_1 &= -x_1 e^{x_2} + x_2 \\
\dot{x}_2 &= x_1 + 3x_2 - x_1^2 \cdot x_2 + u
\end{align*}
\]

*If you want to stabilise this system at the origin, what are the methods you are going to employ?*
Note that

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} = \begin{bmatrix}
-e^{x_2} & 0 \\
0 & -x_1^2 
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 
\end{bmatrix} + \begin{bmatrix} 0 \\
1 
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 
\end{bmatrix} + \begin{bmatrix} 0 \\
1 
\end{bmatrix} u
\]

= \alpha I x + (A + BL)x + Bu

Consequently, a linear controller of the form \( u(x) = k_1 x_1 + k_2 x_2 \) will suffice.

Also,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} = \begin{bmatrix}
-e^{x_2} & 1 \\
0 & -x_1^2 
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 
\end{bmatrix} + \begin{bmatrix} 0 \\
0 
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 
\end{bmatrix} + \begin{bmatrix} 0 \\
1 
\end{bmatrix} u
\]

= \mathbf{J}_2 x + BL + Bu
Consider the single input linear system

\[ \dot{x} = Fx + Gu \]

together with the linear single output equation

\[ y = Hx \]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \), \( y \in \mathbb{R} \), and \( F \in \mathbb{R}^{n \times n} \), \( G \in \mathbb{R}^{n \times 1} \) and \( H \in \mathbb{R}^{1 \times n} \) are constant matrices.

We assume that the pairs \( (F, H) \) and \( (F, G) \) are observable and controllable respectively.
Some definitions

Definition 1. Let \((F, G)\) and \((F, \overline{G})\), where \(F, \overline{F} \in R^{nxn}\) and \(G, \overline{G} \in R^{nx1}\), be two controllable pairs. Then, the pair \((\overline{F}, \overline{G})\) is said to be an equivalent controllable pair of \((F, G)\) if there exists a nonsingular matrix \(P_c\) such that

\[
\overline{F} = P_cFP_c^{-1} \quad \text{and} \quad \overline{G} = P_cG
\]

Lemma 2. Let \((F, G)\) and \((F, \overline{G})\), with \(F, \overline{F} \in R^{nxn}\) and \(G, \overline{G} \in R^{nx1}\), be two equivalent controllable pairs. Then, the matrix \(P_c\) such that \(\overline{F} = P_cFP_c^{-1}\) and \(\overline{G} = P_cG\) is given by \(P_c = Y_cU_c^{-1}\) where \(U_c = [G, FG, \ldots, F^{n-1}G]\) and \(Y_c = [\overline{G}, \overline{FG}, \ldots, \overline{F}^{n-1}\overline{G}]\) are the controllability matrices of the pair \((F, G)\) and \((\overline{F}, \overline{G})\) respectively.
Notations

We denote by:

\[
(F, G) \xrightarrow{P_{c}=Y_{c}U_{c}^{-1}} (\bar{F}, \bar{G})
\]

the transformation of the controllable pair \((F, G)\) into \((\bar{F}, \bar{G})\) via the similarity matrix \(P_{c}\) which is given by \(P_{c}=Y_{c}U_{c}^{-1}\) where \(U_{c}\) is the controllability matrix of \((F, G)\) and \(Y_{c}\) is the controllability matrix of \((\bar{F}, \bar{G})\).
Some remarks

1) Note the formula $P_c = Y_c U_c^{-1}$ are not usable unless the matrices $\bar{F}$, $\bar{G}$ and $\bar{H}$ are fully known in advance.

2) In control systems engineering it is customary to seek for a particular structure of the image pairs $(\bar{F}, \bar{G})$ and $(\bar{F}, \bar{H})$ so that it facilitates the design of a controller or an observer.

3) Generally, the desired forms are either the Brunowski observable and controllable pairs $(A_o, C)$ and $(A_c, B)$ given by:

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ l_1 & l_2 & \cdots & l_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A_o = \begin{bmatrix} k_1 & 1 & \cdots & 0 \\ k_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ k_n & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$
Further remarks

1) The $k_i$'s and $l_i$'s are the coefficients of the characteristic equation of the matrix $F$ under consideration (or that of $A_o$ or $A_c$). As a result, one can readily apply the formulae $P_o = W_o^{-1} Y_o$ and $P_c = Y_c U_c^{-1}$ in such cases.

2) Note that the matrix $A_o$ can be further decomposed as $A_o = A + KC$ where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$ 

Similarly, $A_c$ can be decomposed as $A_c = A + BL$ where $L = (l_1 \; l_2 \; \cdots \; l_n)$.

3) The matrix $A$ is nothing more than a Jordan block with its diagonal entries 0.
The extended Jordan forms

In this work, we search for the transformations that allow a controllable pair \((F, G)\) and an observable pair \((F, H)\) to be transformed into the following pairs \((J_c, B)\) and \((J_o, C)\) where

\[
J_c = \begin{pmatrix}
\alpha_1 & 1 & \cdots & 0 \\
0 & \alpha_2 & \cdots & \vdots \\
0 & \cdots & \ddots & 1 \\
\omega_1 & \omega_2 & \cdots & \omega_n + \alpha_n
\end{pmatrix}, \quad J_o = \begin{pmatrix}
\gamma_1 + \alpha_1 & 1 & \cdots & 0 \\
\gamma_2 & \alpha_2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
\gamma_n & 0 & 0 & \alpha_n
\end{pmatrix},
\]

and where the entries \(\alpha_i, \gamma_i \quad i = 1, \ldots, n\) are chosen arbitrarily; i.e the \(\alpha_i\)'s are not necessarily the eigenvalues of \(F\). The matrices \(C\) and \(B\) are as above. The pairs \((J_o, C)\) and \((J_c, B)\) will be referred to as the Extended Jordan Observable.
Some remarks

Note that the matrix $J_o$ and $J_c$ can be respectively decomposed as $J_o = J_n + \Gamma C$

and $J_c = J_n + B\Omega$ where

$$J_n = \begin{pmatrix}
\alpha_1 & 1 & \cdots & 0 \\
0 & \alpha_2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & \alpha_n
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_n
\end{pmatrix}, \quad \Omega = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_n
\end{pmatrix}.$$

It is important to note that the pairs $(J_o, C)$ and $(J_c, B)$ are indeed observable and controllable pairs respectively since their corresponding observability and controllability matrices are both of rank $n$. 
Further remarks

1) Note that since the $\alpha_i$'s are arbitrary, the entries $\omega_1, \ldots, \omega_n$ and $\gamma_1, \ldots, \gamma_n$ will depend on the choice of the $\alpha_i$'s.

2) In contrast to the Brunowski case, they are not, in general, the coefficients of the characteristic equation of the matrix $F$ or that of $J_o$ or $J_c$.

3) Consequently, for a particular controllable pair $(F, G)$ and a specific choice of $\alpha_1, \ldots, \alpha_n$, the vector $\Omega$ cannot be known in advanced.

4) In addition, it will be very difficult (even though not impossible) to compute the $\omega_i$'s by just equating the characteristig equations of $F$ with that of $J_c$.

5) The same difficulty occurs with the observable pair $(F, H)$ for which it is very difficult to know $\Gamma$ in advanced for a particular choice of $\alpha_i$; $i = 1, \ldots, n$. 
Further remarks

If we wish to apply the formula \( P_c = Y_c U_c^{-1} \) in order to transform \((F, G)\) into \((J_c, B)\)
then we would require the \textit{a priori} knowledge of the vector \( \Omega \).

\[
(F, G)|_{U_c}^{P_c = Y_c U_c^{-1}} \rightarrow (J_c = J_n + B\Omega, C)|_{Y_c}
\]

This suggest that some methods for deducing the vector \( \Gamma \) need to be derived in
order to apply the formula.
Remark

Note that, for example in the controllable case, we can equate the characteristic equation of $F$ and $J_n+B\Omega$ to find $\Omega$. In this case we will have $n$ equations to solve.

For example, for $n=3$, we will have to equate:

$$det[sI-F] = s^3 + a_3s^2 + a_2s + a_1$$

$$det[sI-(J_n+B\Omega)] = s^3 + (-\alpha_2-\omega_3-\alpha_1-\alpha_3)s^2 + (-\omega_2+\alpha_2\omega_3+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\omega_3+\alpha_1\alpha_3)s$$

$$+(-\omega_1+\alpha_1\omega_2+\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\omega_3)$$

That is, we will have to solve:

$$a_3 = -\alpha_2-\omega_3-\alpha_1-\alpha_3$$

$$a_2 = -\omega_2 + \alpha_2\omega_3 + \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\omega_3 + \alpha_1\alpha_3$$

$$a_1 = \omega_1 + \alpha_1\omega_2 + \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\omega_3$$
The main objective

In this work, we shall derive two algorithms that will permit to obtain the transformations $P_c$ and $P_o$ such that

$$ c \rightarrow (F, G)_{P_c}^{{P_c}^{-1}Y_{U_c}} \rightarrow (J_c, B)_{U_c} $$

and

$$ o \rightarrow (F, H)_{P_o}^{{P_o}^{-1}Y_{U_o}} \rightarrow (J_o, C)_{U_o} $$

without the prior knowledge of the vectors $\Gamma$ and $\Omega$.

The proposed algorithms can also be used to transform a controllable and observable pair into their corresponding Brunovski forms ($\alpha_i = 0$).
Remark

The special case \( \alpha_i = \alpha \quad \text{for} \quad i = 1, \ldots, n \), was studied in

K. Busawon, "Control design using Jordan controllable canonical form", *Proc. of the IEEE Conf. on Decision and Control*, Sydney, Australia, 2000, 3386-3392.


In addition, no justifications were provided as to why such algorithms work.
Lemma 3. Let \((F, G)\), with \(F \in \mathbb{R}^{n \times n}\) and \(G \in \mathbb{R}^{n \times 1}\), be a controllable pair. Let 

\[ U_c = [G, FG, \ldots, F^{n-1}G] \]

be its controllability matrix. Then,

\[
(U_c^{-1}FU_c)^T = A + BL
\]

where \(L = (-a_0, -a_1, \ldots, -a_{n-1})\) with \(a_i\), \(i = 1, \ldots, n-1\) are the coefficients of the characteristics polynomial of \(F\); that is \(p(s) = s^n + a_{n-1}s^{n-1} \cdots + a_1s + a_0\).

In addition, \(V_cU_c^{-1}G = B\) where \(V_c = [B, AB, \ldots, A^{n-1}B] = \begin{pmatrix}
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}\).
Algorithm 1. Transformation into the controllable Brunowski form without prior knowledge of $L$

Consider the transformation $(F,G)_{U_c}^{p=U_c^{-1}} ightarrow (A+BL,B)_{U_c}$

The algorithm for the computation of $P_c$ is given as follows:

Step 1. Compute $U_c = [G, FG, \ldots, F^{n-1}G]$

Step 2. Next, compute $(U_c^{-1}FU_c)^T$. (This is equal to $A+BL$ according to Lemma 3).

Step 3. Then, compute $Y_c = [B, (U_c^{-1}FU_c)^T B, \ldots, ((U_c^{-1}FU_c)^T)^{n-1} B]$.

Step 4. Finally, $P_c = Y_c U_c^{-1}$.
Transformation into the EJC form

We shall now derive an algorithm that will permit to transform \((F,G)\) into \((J_n + B\Omega, B)\).

Note that since we know how to transform \((F,G)\) into \((A + BL, B)\). We only have to find a transformation \(P_j\) such that

\[
\left.\left(A + BL, B\right)\right|_{Y_z}^{P_j = Z_c Y^{-1}_c} \rightarrow \left.\left(J_n + B\Omega, B\right)\right|_{Z_c}
\]

where \(Y_z = [B, (A + BL)B, \ldots, (A + BL)^{n-1} B]\) and \(Z_c = [B, J_c B, \ldots, J_c^{n-1} B]\).

Note that here we do not know \(\Omega\) in advanced. Therefore, we cannot use \(P_j = Z_c Y^{-1}_c\) in order to derive \(P_j\) directly. Therefore, we need to derive an algorithm to find \(\Omega\) step by step and then apply the formula \(P_j = Z_c Y^{-1}_c\).
Methodology

• For this, we shall first show that any Brunowski controllable pair \((A + BL, B)\) can be transformed into the pair \((J_n + \Gamma C, B)\) for some vector \(\Gamma\) that we will develop.

• Next, we shall show that the pair \((J_n + \Gamma C, B)\) is transformable into the pair \((J_n + B\Omega, B)\) provided that \(\Gamma\) and \(\Omega\) satisfy certain condition that we shall detail.
**Condition for the equivalence of** \((A+BL,B)\) **and** \((J_n + \Gamma C,B)\)

The next lemma shows that any Brunowski controllable pair \((A+BL,B)\) is equivalent to \((J_n + \Gamma C,B)\).

**Lemma 5.** *The pair* \((A+BL,B)\) *is equivalent to* \((J_n + \Gamma C,B)\) *if* \(\Gamma = W_c(L - L_0)^T\) *where*

\[W_c\] *is the controllability matrix of the pair* \((J_n,B)\) *and*

\[L_0 = B^T\left[(W_c^{-1}J_nW_c)^T - A\right]\]

\[W_c = [B, J_nB, \ldots, J_n^{n-1}B].\]
Condition for the equivalence of \((J_n + \Gamma C, B)\) and \((J_n + B\Omega, B)\)

We now consider the problem of finding the conditions under which the controllable pairs \((J_n + \Gamma C, B)\) and \((J_n + B\Omega, B)\) are equivalent. More precisely, we would like to know how to choose \(\Gamma\) and \(\Omega\) such that the two pairs are equivalent.

The following lemma answers this question:

**Lemma 6.** The controllable pair \((J_n + \Gamma C, B)\) is equivalent to \((J_n + B\Omega, B)\) if

\[
\Gamma = W_c (\Theta_o^{-1})^T \Omega^T \quad \text{where} \quad W_c = [B, J_n B, \ldots, J_n^{n-1} B] \quad \text{and} \quad \Theta_o = \begin{pmatrix} C \\ CJ_n \\ \vdots \\ CJ_n^{n-1} \end{pmatrix}.
\]
Algorithm 2: Transformation of \((F,G)\) into the EJC form

We can now derive an algorithm to transform a controllable pair \((F,G)\) into \((J_n + B\Omega, B)\) without the prior knowledge of the vector \(\Omega\).

This transformation is done in two stages, as follows

\[
\begin{align*}
(F,G) & \quad \overset{p = Y_U^{-1}}{\longrightarrow} \quad (A + BL, B) \\
(A + BL, B) & \quad \overset{p = Z_U^{-1}}{\longrightarrow} \quad (J_n + B\Omega, B)
\end{align*}
\]

so that the overall transformation is given by: \(P = Z_U^{-1}U\).
Algorithm: Consequently, the algorithm is given as:

**Step 1.** Compute \( U_c = [G, FG, \ldots, F^{n-1}G] \) and \( U_c^{-1} \).

**Step 2.** Next, compute \( L = B^T \left( U_c^{-1}FU_c \right)^T - A \).

**Step 3.** Then, compute \( W_c = [B, J_nB, \ldots, J_n^{n-1}B] \).

**Step 4.** Calculate \( L_0 = B^T \left( W_c^{-1}J_nW_c \right)^T - A \).

**Step 5.** Compute \( \Theta = \begin{pmatrix} C \\ CJ_n \\ \vdots \\ CJ_n^{n-1} \end{pmatrix} \).

**Step 6.** Compute \( \Omega = (L - L_0)\Theta \).

**Step 7.** Define \( J_c = J_n + B\Omega \).

**Step 8.** Compute \( Z_c = [B, J_cB, \ldots, J_c^{n-1}B] \).

**Step 9.** Finally, compute \( P_c = Z_cU_c^{-1} \).
Remarks:

i) If we consider an n-dimensional square matrix \( F \) which have an eigenvalue \( \lambda \) with multiplicity order \( n \) and assume that we can find an \( n \times 1 \) matrix \( G \) such that the pair \((F,G)\) is controllable; then if we choose, \( \alpha_i = \lambda \) for \( i=1,\ldots,n \), then transformation \( P_j = Z_c Y_c^{-1} \) will be equal to the standard Jordan transformation of the matrix \( F \).

ii) If \( J_n = A \); that is \( \alpha_1 = \ldots \alpha_n = 0 \), then \( L_0 = B^T \left( W_c^{-1} J_n W_c \right)^T - A \) = 0. Consequently, \( \Omega = L \).
Transformation of \((F,H)\) into \((J_n + \Gamma C, C)\)

We shall give an algorithm to transform an pair \((F,H)\) into \((J_n + \Gamma C, C)\) without the prior knowledge of the vector \(\Gamma\). More precisely, we are concerned with finding

\[ P_o = Z_o^{-1} \gamma_o \text{ such that } (F,H)_{\gamma_o}^{P_o=Z_o^{-1} \gamma_o} \rightarrow (J_n + \Gamma C, C)_{Z_o} \]
Algorithm 3: EJOC form

Step 1. Compute \( Y_o = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{pmatrix} \) and \( Y_o^{-1} \).

Step 2. Next, compute \( L = B^T [Y_o F Y_o^{-1} - A] \).

Step 3. Then, compute \( \Theta_o = \begin{pmatrix} C \\ CJ_n \\ \vdots \\ CJ_n^{n-1} \end{pmatrix} \).

Step 4. Calculate \( L_0 = B^T [\Theta_o J_n \Theta_o^{-1} - A] \).

Step 5. Compute \( W_c = [B, J_n B, \ldots, J_n^{n-1} B] \).

Step 6. Compute \( \Gamma = W_c (L - L_0)^T \).

Step 7. Define \( J_o = J_n + \Gamma C \).

Step 8. Compute \( Z_o = \begin{pmatrix} C \\ CJ_c \\ \vdots \\ CJ_c^{n-1} \end{pmatrix} \).

Step 9. Finally, compute \( P_o = Z_o^{-1} Y_o \).
Example 1- EJCC form

Consider the following matrix controllable pair: \[ F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ -2 & 0 & 2 \end{pmatrix}, \quad G = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \]

Suppose we want to transform \((F, G)\) into the EJCC form with \(\alpha_1 = 2, \quad \alpha_2 = 3\) and \(\alpha_2 = 5\). Following all the steps in Algorithm 2, one can show that \(L = (8 \quad -11 \quad 6)\),

\[ L_0 = (30 \quad -31 \quad 10) \quad \Omega = (2 \quad 0 \quad -4) \quad \text{and that} \quad P = \begin{pmatrix} \frac{3}{31} & \frac{5}{31} & -\frac{7}{31} \\ \frac{11}{31} & \frac{8}{31} & -\frac{5}{31} \\ -\frac{12}{31} & \frac{11}{31} & -\frac{3}{31} \end{pmatrix}. \]

We can check that \(PcFPc^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & -4 \end{pmatrix} \) and \(PcG = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \)
Example 2- EJO form

Consider the following observable pair \( F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ -2 & 0 & 2 \end{pmatrix} \) and \( H = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \).

Again, suppose we want to transform \( (F, H) \) into the EJO form with \( \alpha_1 = 2, \quad \alpha_2 = 3 \) and \( \alpha_3 = 5 \). Following all the steps in Algorithm 3, we can show that \( \Gamma^T = \begin{pmatrix} -4 & -12 & -22 \end{pmatrix} \)

and \( P_o = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 6 & 3 \\ 4 & 13 & 3 \end{pmatrix} \).

We can check that \( P_o F P_o^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ -12 & 3 & 1 \\ -22 & 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{pmatrix} + \begin{pmatrix} -4 & 0 & 0 \\ -12 & 0 & 0 \\ -22 & 0 & 0 \end{pmatrix} \) and

\( H P_o^{-1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \).
Some applications

1) When designing controllers or observers, we can choose the $\alpha_i$’s to be negative.

$$
J_n = \begin{pmatrix}
\alpha_1 & 1 & \cdots & 0 \\
0 & \alpha_2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & \alpha_n
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_n
\end{pmatrix}, \quad \Omega = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_n
\end{pmatrix}.
$$

2) In special cases, the algorithm can also be used to for state or time dependent systems of the following form

$$\dot{x} = F(t)x + G(t)u \quad \text{or} \quad \dot{x} = F(x)x + G(x)u$$

with $y=H(t)x$ or $y=H(x)x$. Obviously, the transformations will depend on $t$ or $x$.

For example,

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
-e^{x_2} & 1 \\
0 & -x_1^2
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
1 & 3
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
1
\end{pmatrix} u
$$
3) Finally, we know that single-output uniformly observable of the form
\[ \dot{x} = f(x) + g(x)u, \quad y = h(x) \]
can be transformed into \( \dot{z} = Az + \varphi(z)u, \quad y = Cz \) where \( \varphi(z) \) is lower triangular.

4) We can safely say that single-output uniformly observable of the form
\[ \dot{x} = f(x) + g(x)u, \quad y = h(x) \]
 can be transformed into \( \dot{z} = J_n z + \psi(z)u, \quad y = Cz \) where \( \psi(z) \) is lower triangular.

5) Finally, if we set
\[
\Lambda = \text{diag} \left( 1, \frac{1}{\beta_1}, \frac{1}{\beta_1 \beta_2}, \frac{1}{\beta_1 \beta_2 \beta_3}, \ldots, \frac{1}{\beta_1 \beta_2 \ldots \beta_{n-1}} \right)
\]
then we can show that
\[
\Lambda J_n \Lambda^{-1} = \begin{pmatrix} \alpha_1 & \beta_1 & \ldots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \beta_{n-1} \\ 0 & \ldots & 0 & \alpha_n \end{pmatrix}.
\]
Conclusion

1) we have proposed an algorithm that permits to derive a transformation allowing to transform a controllable pair into the Extended Jordan Controllable Canonical form.

2) Next, by duality we have derived an algorithm that permits to transform an observable pair into the Extended Jordan Observable Canonical form.

3) Detailed justification of the proposed algorithms are provided and some examples are given to show the validity of the algorithms.

4) Even though not shown here, the algorithms can be further developed and used as a framework to transform a given observable and controllable pair into other canonical observable and controllable pairs respectively.